Ordinary Differential Equations

Exercise Sheet 2

Exercise 1. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Show that the solution of the IVP:*

$$\begin{cases} y' = g(y), & y(t_0) = y_0 \\ z' = f(y)z, & z(t_0) = z_0 \end{cases}$$

is global in $\mathbb{R}$.

[Hint: Use exercise 7 of exercise sheet 1.]

Exercise 2. *Convert the following equations/system to first order systems:*

(i) $y''' + (y'')^2 + e^t y y' - 2(\sin t)y^4 = 0$

(ii) $y^{(4)} = y^2(y')^2 + y'''y'' - e^{2t}y^{(4)}$

(iii)

$$\begin{cases} y^{(4)} + (z'')^2 y^3 - t^{10} y'' z + z^3 (y')^4 = 0 \\ z''' = (y''')^2 - e^{2t}(z')^2 + y'' z^5 \end{cases}$$

Exercise 3. *Compute the Wronskian of*

(i) y^a, y^b, y^c, *where* $y > 0$ *and* $a, b, c \in \mathbb{R}$.

(ii) $y^m \sin \log y^n$, $y^m \cos \log y^n$, *where* $y > 0$ *and* $m, n \in \mathbb{N}$.

Are the functions in each case linearly independent?

Exercise 4. *Find the general solution to the equations:*

(i) $y^{(4)} - 5y'' + 4y = e^t - te^{2t}$

(ii) $y'' - 3y' + 2y = 14 \sin 2t - 18 \cos 2t$

Exercise 5. *Solve the IVPs:*

(i) $y''' = y$, $y(0) = 1$, $y'(0) = y''(0) = 0$

(ii) $y''' + y'' = t + e^{-t}$, $y(0) = y''(0) = 1$, $y'(0) = 0$

Exercise 6. *Compute the exponential matrices* e^{At}, e^{Bt} *for*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

Does $e^{At}e^{Bt} = e^{Bt}e^{At}$ *hold?*

Exercise 7. *Find the general solution of the system* $\vec{y}' = A\vec{y}$, *where*

$$A = \begin{bmatrix} 2 & -5 & 0 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$

and compute the Wronskian of a basis of solutions.

Exercise 8. *Write the system*

$$\begin{cases} y_1' = y_1 - 2y_3 \\ y_2' = y_2 \\ y_3' = y_1 - y_2 - y_3 \end{cases}$$

in matrix form and compute its general solution.

Exercise 9. *Solve the IVP:*

$$\vec{y}' = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \vec{y}, \qquad \vec{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$